ECS455: Chapter 5

OFDM

5.4 Cyclic Prefix (CP)

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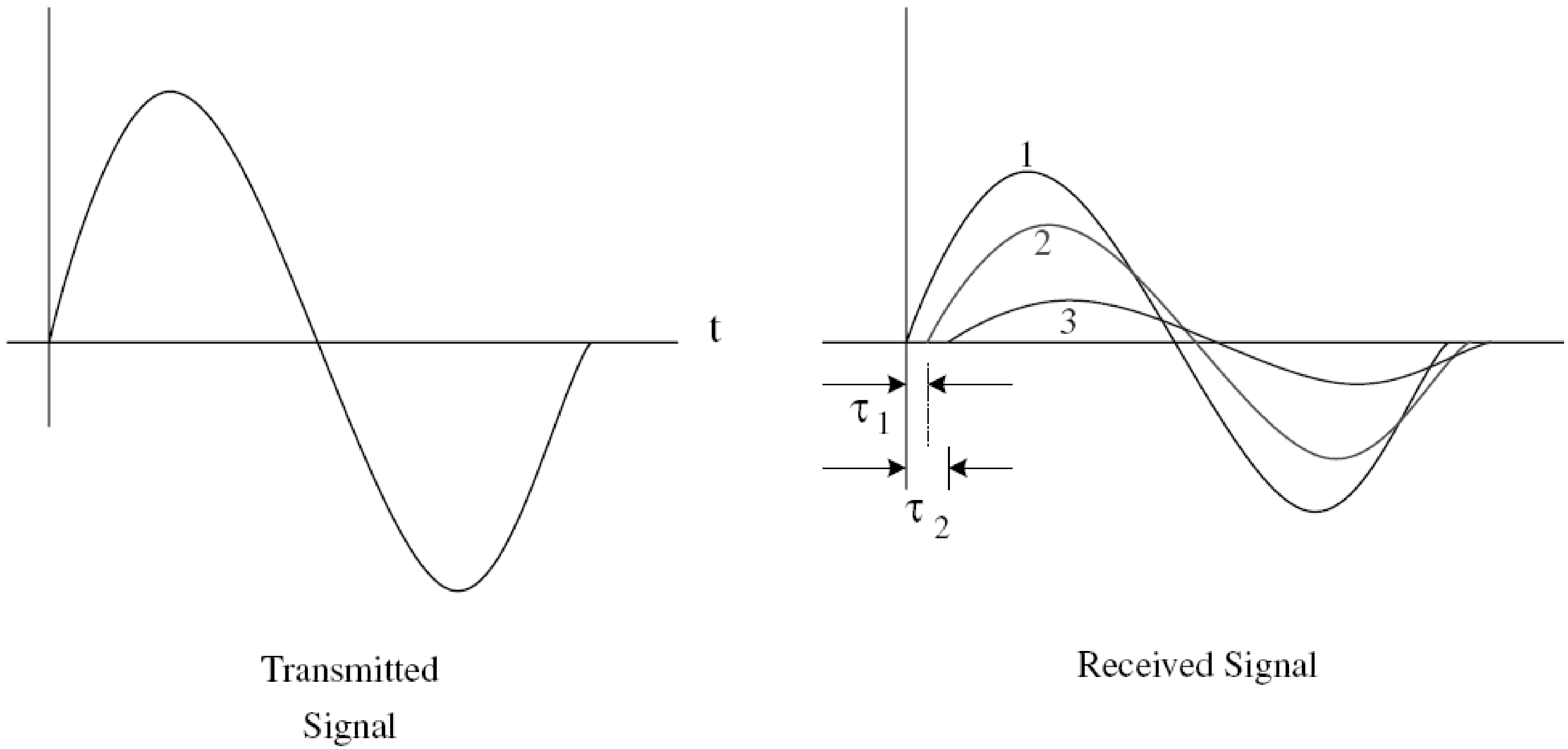
Three steps towards modern OFDM

1. Mitigate Multipath (ISI): Decrease the rate of the original data stream via multicarrier modulation (FDM)
2. Gain Spectral Efficiency: Utilize orthogonality
3. Achieve Efficient Implementation: FFT and IFFT

- Extra step: Completely eliminate ISI and ICI
  - Cyclic prefix

Cyclic Prefix: Motivation (1)

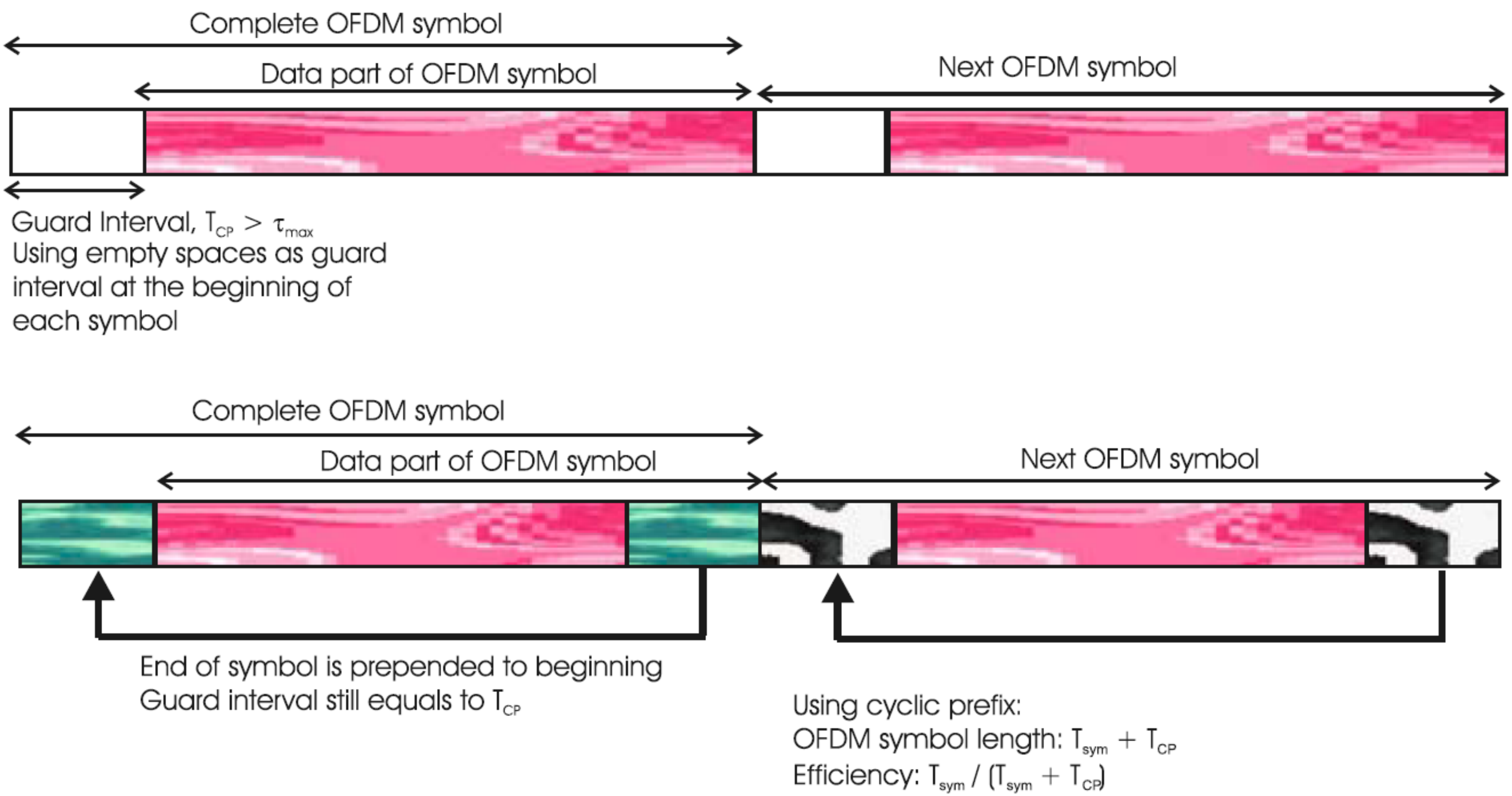
- Recall: Multipath Fading and Delay Spread



Cyclic Prefix: Motivation (2)

- OFDM uses large symbol duration T_s
  - compared to the duration of the impulse response $\tau_{\max}$ of the channel
  - to reduce the amount of ISI
- **Q**: Can we "eliminate" the multipath **(ISI)** problem?
- **A**: To reduce the ISI, add **guard interval** larger than that of the estimated delay spread.
- If the guard interval is left empty, the orthogonality of the sub-carriers no longer holds, i.e., **ICI** (inter-channel interference) still exists.
- **Solution**: To prevent **both** the **ISI** as well as the **ICI**, OFDM symbol is **cyclically extended** into the guard interval.

Cyclic Prefix

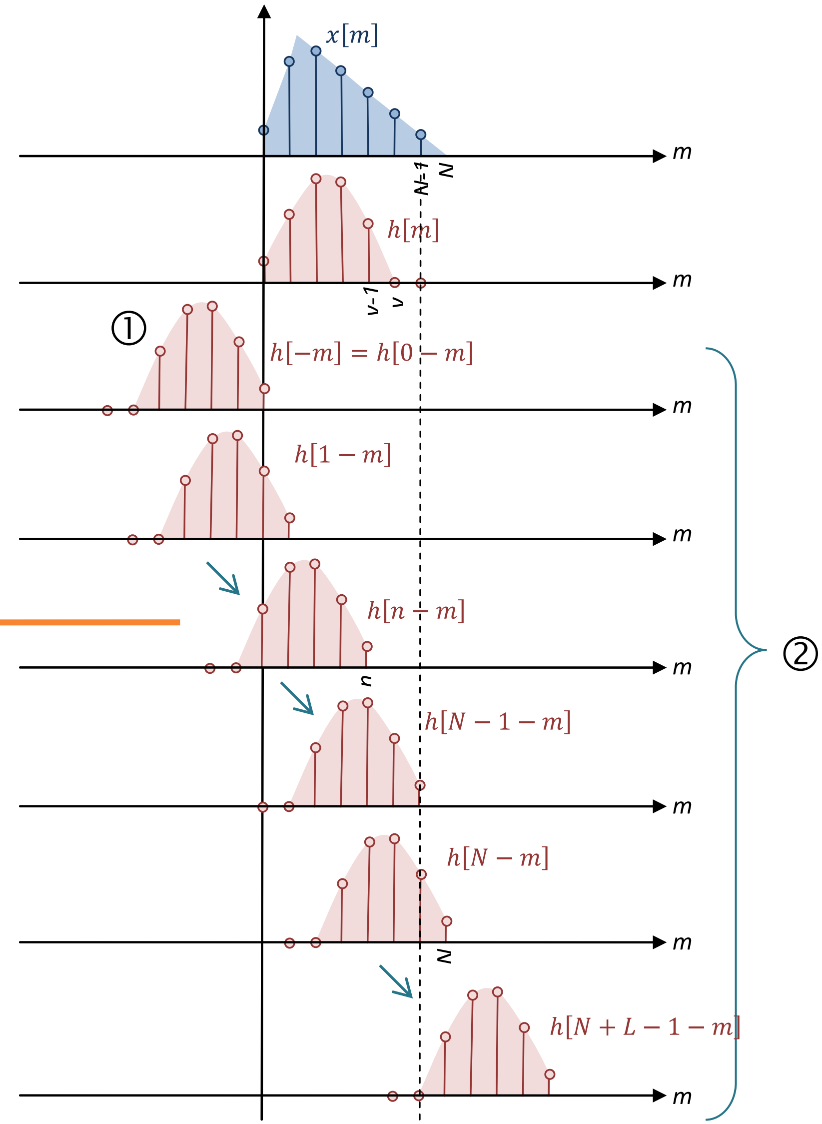


Recall: Convolution

① Flip
② Shift
③ Multiply (pointwise)
④ Add

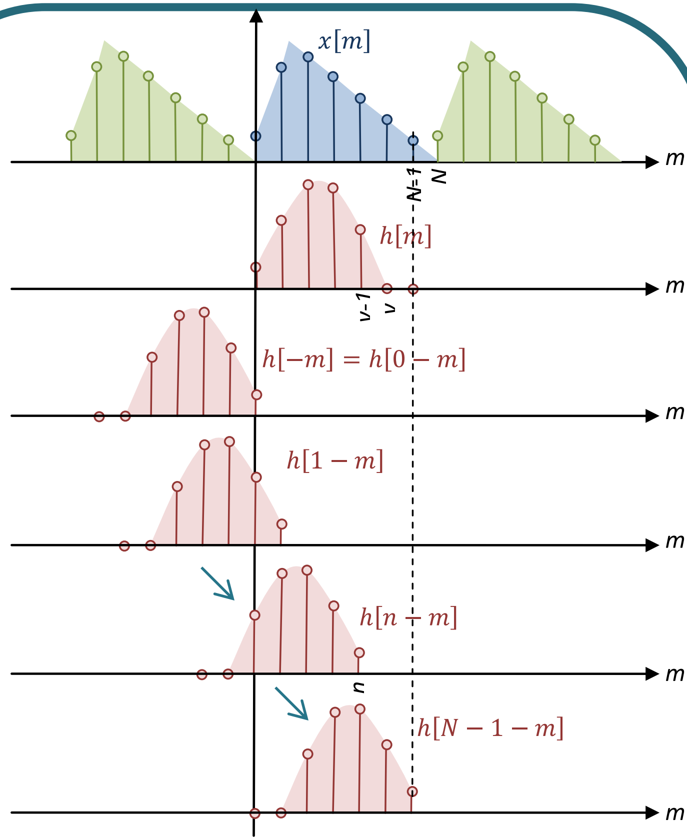
④ Σ ⊗ ③

$$\{x*h\}[n]=\sum_m x[m]h[n-m]$$



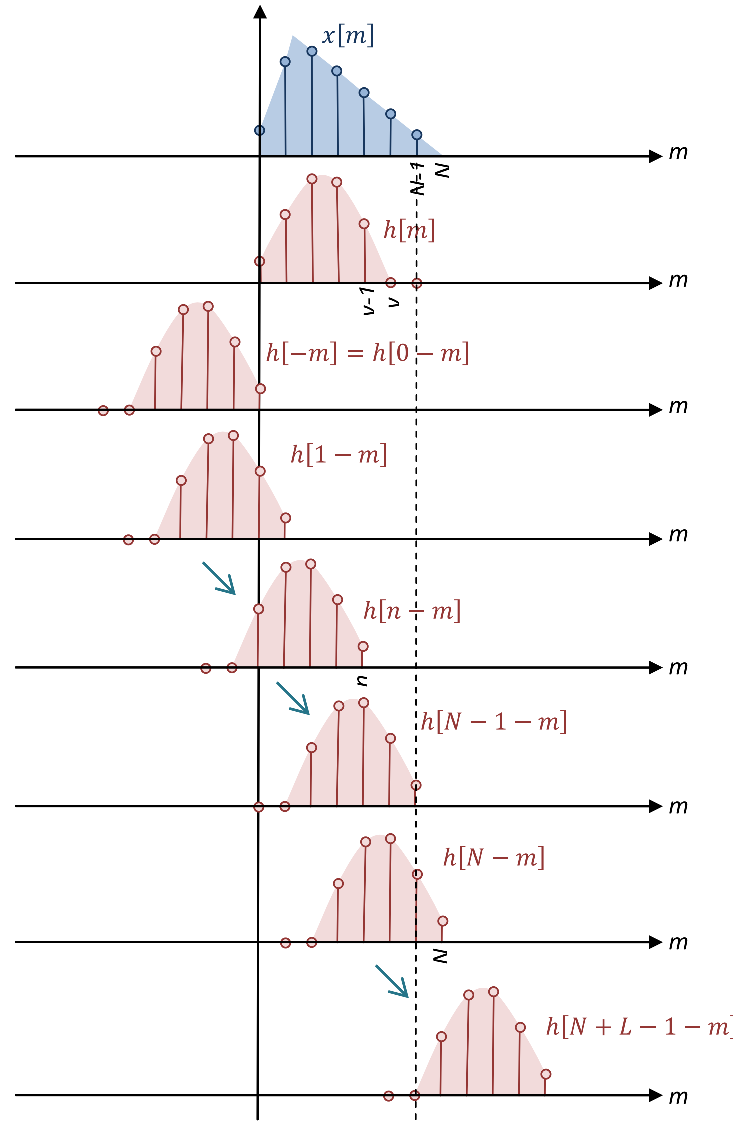
N-pt signal ⊛ N-pt signal = N-pt signal

Circular Convolution



Replicate *x* (now it looks periodic)
Then, perform the usual convolution
only on n = 0 to *N*-1

(Regular Convolution)



Circular Convolution: Examples 1

Find

x h

$$[1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

$= [4 \quad 13 \quad 28 \quad 27 \quad 18]$

1 2 3
6 5 4
6 5 4
6 5 4
6 5 4
6 5 4

4
$5 \times 1 + 4 \times 2 = 13$
$6 \times 1 + 5 \times 2 + 4 \times 3 = 28$
$6 \times 2 + 5 \times 3 = 27$
18

$$[1 \quad 2 \quad 3] \circledast [4 \quad 5 \quad 6]$$

$= [31 \quad 31 \quad 28]$

. . 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 . . .
6 5 4
6 5 4
6 5 4

$12 + 15 + 4 = 31$
$18 + 5 + 8 = 31$
$6 + 10 + 12 = 28$

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0]$$

$= [4 \quad 13 \quad 28 \quad 27 \quad 18]$

Discussion

- ***Regular convolution*** of an N_1–point vector and an N_2–point vector gives (N_1+N_2-1)-point vector.
- ***Circular convolution*** is performed between two equal-length vectors. The results also has the same length.
- Circular convolution can be used to find the regular convolution by **zero-padding**.
  - Zero-pad the vectors so that their length is N_1+N_2-1.
  - Example:

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0] = [1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

- In modern OFDM, we want to perform circular convolution via regular convolution.

Circular Convolution in Communication

- We want the receiver to obtain the circular convolution of the signal (channel input) and the channel.
- Q: Why?
- A:
  - **CTFT**: **convolution** in time domain corresponds to **multiplication** in frequency domain.
    - This fact does not hold for DFT.
  - **DFT**: circular **convolution** in (discrete) time domain corresponds to **multiplication** in (discrete) frequency domain.
    - We want to have multiplication in frequency domain.
    - So, we want circular convolution and not the regular convolution.
- Problem: Real channel does regular convolution.
- Solution: With **cyclic prefix**, regular convolution can be used to create circular convolution.

Example 2

Let's look closer at how we carry out the circular convolution operation. Recall that we replicate the x and then perform the regular convolution (for N points)

$x[n]$ $\circledast$ $h[n]$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = ?$$

Solution:

$$\begin{array}{ccccccccccccccc}
1 & -2 & 3 & 1 & 2 & 1 & -2 & 3 & 1 & 2 & 1 & -2 & 3 & 1 & 2 \\
 & 0 & 0 & 1 & 2 & 3 & & & & & & & & & \\
 & & 0 & 0 & 1 & 2 & 3 & & & & & & & & \\
 & & & 0 & 0 & 1 & 2 & 3 & & & & & & & \\
 & & & & 0 & 0 & 1 & 2 & 3 & & & & & & \\
 & & & & & 0 & 0 & 1 & 2 & 3 & & & & &
\end{array}$$

$$1\times1+2\times2+1\times3=1+4+3=8$$

$$2\times1+1\times2+(-2)\times3=2+2-6=-2$$

$$1\times1+(-2)\times2+3\times3=1-4+9=6$$

$$(-2)\times1+3\times2+1\times3=-2+6+3=7$$

$$3\times1+1\times2+2\times3=3+2+6=11$$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = \underbrace{[8 \quad -2 \quad 6 \quad 7 \quad 11]}_{\text{Goal: Get these numbers using regular convolution}}$$

Example 2

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = ?$$

Observation: We don't need to replicate the x indefinitely. Furthermore, when h is shorter than x, we need only a part of one replica.

Not needed in the calculation

1	−2	3	1	2	1	−2	3	1	2	1	−2	3	1	2
	0	0	1	2	3									
		0	0	1	2	3								
			0	0	1	2	3							
				0	0	1	2	3						
					0	0	1	2	3					

$$1\times1+2\times2+1\times3=1+4+3=8$$
$$2\times1+1\times2+(-2)\times3=2+2-6=-2$$
$$1\times1+(-2)\times2+3\times3=1-4+9=6$$
$$(-2)\times1+3\times2+1\times3=-2+6+3=7$$
$$3\times1+1\times2+2\times3=3+2+6=11$$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = [8 \quad -2 \quad 6 \quad 7 \quad 11]$$

Example 2

Try this: use only the necessary part of the replica and then convolute (regular convolution) with the channel.

$$\underbrace{[1 \quad 2}\quad 1 \quad -2 \quad 3 \quad 1 \quad 2] * [3 \quad 2 \quad 1] = ?$$

Copy the **last ν samples** of the symbols at the **beginning** of the symbol. This partial replica is called the **cyclic prefix**.

$$\overbrace{1 \quad 2} \quad 1 \quad -2 \quad 3 \quad 1 \quad 2$$

1	2	3								$1\times 3 = 3$
	1	2	3							$1\times 2 + 2\times 3 = 2+6 = 8$
		1	2	3						$1\times 1 + 2\times 2 + 1\times 3 = 1+4+3 = 8$
			1	2	3					$2\times 1 + 1\times 2 + (-2)\times 3 = 2+2-6 = -2$
				1	2	3				$1\times 1 + (-2)\times 2 + 3\times 3 = 1-4+9 = 6$
					1	2	3			$(-2)\times 1 + 3\times 2 + 1\times 3 = -2+6+3 = 7$
						1	2	3		$3\times 1 + 1\times 2 + 2\times 3 = 3+2+6 = 11$
							1	2	3	$1\times 1 + 2\times 2 = 1+4 = 5$
								1	2 3	$2\times 1 = 2$

Junk!

Example 2

- We now know that

$$\underbrace{[1 \quad 2}_{\text{Cyclic Prefix}} \quad 1 \quad -2 \quad 3 \quad 1 \quad 2] * [3 \quad 2 \quad 1] = [3 \quad 8 \quad \underbrace{8 \quad -2 \quad 6 \quad 7 \quad 11}_{[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0]} \quad 5 \quad 2]$$

- Similarly, you may check that

$$\underbrace{[-2 \quad 1}_{\text{Cyclic Prefix}} \quad 2 \quad 1 \quad -3 \quad -2 \quad 1] * [3 \quad 2 \quad 1] = [-6 \quad -1 \quad \underbrace{6 \quad 8 \quad -5 \quad -11 \quad -4}_{[2 \quad 1 \quad -3 \quad -2 \quad 1] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0]} \quad 0 \quad 1]$$

Example 3

- We know, from Example 2, that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
```

And that

```
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Check that

```
 [ 1 2 1 -2 3  1  2  0  0 0 0  0   0  0] * [3 2 1]
= [ 3 8 8 -2 6  7 11  5  2 0 0  0   0  0 0 0]
```

and

```
 [ 0 0 0  0 0  0  0 -2  1 2 1 -3  -2  1] * [3 2 1]
= [ 0 0 0  0 0  0  0 -6 -1 6 8 -5 -11 -4 0 1]
```

Example 4

- We know that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Using Example 3, we have

```
    [ 1 2 1 -2 3  1  2 -2  1 2 1 -3  -2  1] * [3 2 1]

= ( [ 1 2 1 -2 3  1  2  0  0 0 0  0   0  0]
   +[ 0 0 0  0 0  0  0 -2  1 2 1 -3  -2  1] ) * [3 2 1]

=   [ 3 8 8 -2 6  7 11  5  2 0 0  0   0  0 0 0]
   +[ 0 0 0  0 0  0  0 -6 -1 6 8 -5 -11 -4 0 1]

=   [ 3 8 8 -2 6  7 11 -1  1 6 8 -5 -11 -4 0 1]
```

Putting results together…

- Suppose $\underline{x}^{(1)} = [1\ -2\ 3\ 1\ 2]$ and $\underline{x}^{(2)} = [2\ 1\ -3\ -2\ 1]$
- Suppose $\underline{h} = [3\ 2\ 1]$
- At the receiver, we want to get
  - $[1\ -2\ 3\ 1\ 2] \circledast [3\ 2\ 1\ 0\ 0] = [8\ -2\ 6\ 7\ 11]$
  - $[2\ 1\ -3\ -2\ 1] \circledast [3\ 2\ 1\ 0\ 0] = [6\ 8\ -5\ -11\ -4]$
- We transmit $[\underbrace{1\ \ 2}_{\text{Cyclic prefix}}\ 1\ -2\ 3\ 1\ 2\ \underbrace{-2\ \ 1}_{\text{Cyclic prefix}}\ 2\ 1\ -3\ -2\ 1]$.
- At the receiver, we get

$$[\ 1\ 2\ 1\ -2\ 3\ 1\ 2\ -2\ 1\ 2\ 1\ -3\ -2\ 1] * [3\ 2\ 1]$$

$$= [\underbrace{3\ \ 8}\ 8\ -2\ 6\ 7\ 11\ \underbrace{-1\ \ 1}\ 6\ 8\ -5\ -11\ -4\ \underbrace{0\ \ 1}]$$

Junk! To be thrown away by the receiver.

Circular Convolution: Key Properties

- Consider an N-point signal $x[n]$
- **Cyclic Prefix (CP) insertion**: If $x[n]$ is extended by copying the last ν samples of the symbols at the beginning of the symbol:

$$\tilde{x}[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ x[n+N], & -\nu \le n \le -1 \end{cases}$$

- Key Property 1:

$$\{h \circledast x\}[n] = (h * \tilde{x})[n] \text{ for } 0 \le n \le N-1$$

- Key Property 2:

$$\{h \circledast x\}[n] \xrightarrow{\text{FFT}} \boxed{H_k X_k}$$

OFDM with CP for Channel w/ Memory

- We want to send N samples $S_0, S_1, \ldots, S_{N-1}$ across noisy channel with memory.
- First apply IFFT: $S_k \xrightarrow{\text{IFFT}} s[n]$
- Then, add cyclic prefix

$$\hat{s} = \left[s[N-\nu], \ldots, s[N-1], s[0], \ldots, s[N-1] \right]$$

- This is inputted to the channel.
- The output is

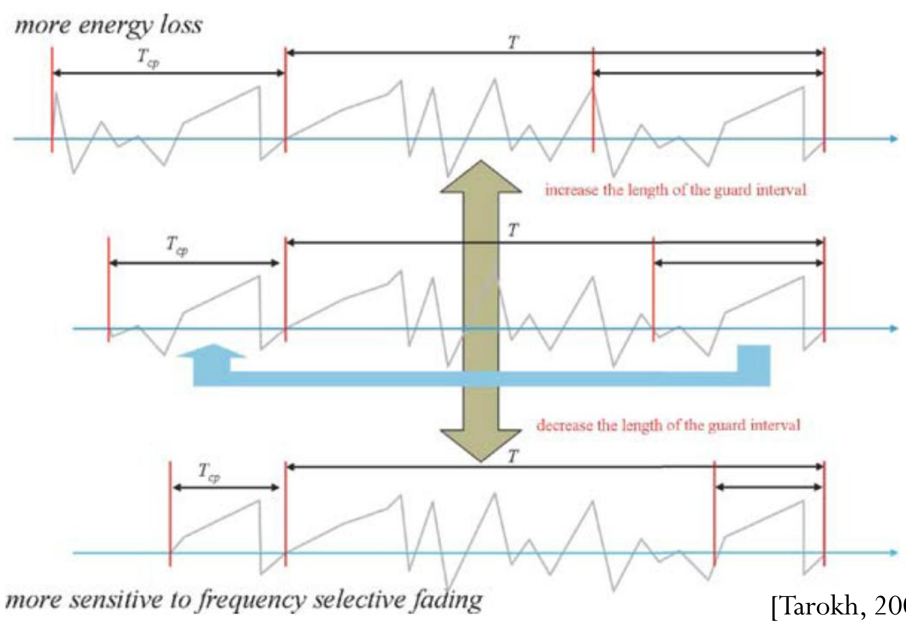
$$y[n] = \left[p[N-\nu], \ldots, p[N-1], r[0], \ldots, r[N-1] \right]$$

- Remove cyclic prefix to get $r[n] = h[n] \circledast s[n] + w[n]$
- Then apply FFT: $r[n] \xrightarrow{\text{FFT}} R_k$
- By circular convolution property of DFT, $R_k = H_k S_k + W_k$

No ICI!

OFDM System Design: CP

- A good ratio between the CP interval and symbol duration should be found, so that all multipaths are resolved and not significant amount of energy is lost due to CP.
- As a thumb rule, the CP interval must be two to four times larger than the root mean square (RMS) delay spread.



[Tarokh, 2009, Fig 2.9]

Summary

- The CP at the beginning of each block has two main functions.
- As guard interval, it prevents contamination of a block by ISI from the previous block.
- It makes the received block appear to be periodic of period N.
  - Turn regular convolution into circular convolution
  - Point-wise multiplication in the frequency domain

Reference

- A. Bahai, B. R. Saltzberg, and M. Ergen, ***Multi-Carrier Digital Communications: Theory and Applications of OFDM***, 2nd ed., New York: Springer Verlag, 2004.